UNITED STATES
NAVAL POSTGRADUATE SCHOOL

SIMPLIFICATION FOR MUTUAL IMPEDANCE OF CERTAIN ANTENNAS

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REPORT NO. 3

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ABSTRACT

The formula for mutual impedance by the generalized circuit method is reduced to a form which greatly reduces the number of integrations required in determining the mutual impedance of various combinations of open wire and terminated wire antennas. The application is illustrated by finding the mutual impedance of the legs of an open wire X-antenna, the latter being equivalent to the radiation impedance of a biconical antenna, and for small angle cones, reduces to a form from which Schelkunoff's inverse terminal impedance may be determined.

GENERALIZED MUTUAL IMPEDANCE

The mutual impedance of two antenna configurations is given in the generalized circuit by,[1]

$$jkZ_{21}/30 = -\oint_1\oint_2 Re[f_1(P_1)^* f_2(P_2)]\ \tilde{\nabla}_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \tag{1}$$

in which

P_1 = any point along the axis of wire *one*

P_2 = any point along the axis of wire *two*

r_{21} = distance from P_1 to P_2

$e(r_{21}) = r_{21}^{-1}\ exp(-jkr_{21})$

$k = \omega(\mu_0\epsilon_0)^{\frac{1}{2}} = 2\pi/\lambda$

$\mu_0 = 4\pi(10)^{-7}$ *henries per meter*

$\epsilon_0 = (36\pi\cdot 10^9)^{-1}$ *farads per meter*

$\tilde{\nabla}_1 = \nabla_1(\nabla_1\cdot\ ) + k^2$ = operator *deltil*, with the subscript indicating the position at which the differentiations are to be performed

$*$ = the complex conjugate to be taken

$f_i(P_i)$ = the current distribution function along a wire

Re = the real part to be taken

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1. J. G. Chaney, 'A critical study of the circuit concept', J. Appl. Phys. 22 12, 1429 (1951).

With the exception of the distances involved in the case of two identical antennas, the integrations to be performed in equation (1) are the same as those to be performed in determing the self radiation impedance by the conventional method of finding the mutual impedance between an axial path and a surface path. Equation (1) has been integrated for the self radiation impedance of a terminated long wire antenna,[2] a terminated Vee-antenna,[3] and a terminated rhombic antenna;[3] also for the mutual impedance of two parallel staggered standing wave antennas of unequal length,[2] two parallel staggered terminated long wire antennas,[2] and for a standing wave antenna parallel staggered with a long wire antenna.[2]

In the case of the rhombic antenna, the algebra was greatly reduced by assuming the axial and surface paths to be interchangeable and by subjecting the currents to equations of constraint. This resulted in requiring the actual evaluation of only three integrals after postulating an unattenuated wave of current travelling along the rhombus. The resulting formula has been verified by the lengthly evaluation of the large number of integrals involved when the current distribution is postulated at the outset.[4] This reduction in algebra suggests that equation (1) be examined for similar reductions when applied in the derivation of formulas for mutual impedances.

Each of the two antennas (Fig.1) will be assumed to be physically symmetric with respect to its generator. Hence,

$$jkZ_{21}/30 = -\left(\oint_e^c+\oint_b^d\right)\left(\oint_f^n+\oint_m^g\right) Re[f_1(P_1)^*f_2(P_2)]\ \tilde{\nabla}_1\ [e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \quad (2)$$

Now let

$$A = -\left(\oint_e^c+\oint_b^d\right)\left(\oint_f^n+\oint_n^g\right) Re[f_1(P_1)^*f_2(P_2)]\ \nabla_1[\nabla_1\cdot e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \quad (3)$$

Recall that

$$\nabla_1\cdot e(r_{21})d\bar{r}_2 = -\nabla_2 e(r_{21})\cdot d\bar{r}_2 \quad (4)$$

and integrate by parts for

$$\begin{aligned} A = {} & \left(\oint_f^n+\oint_m^g\right)\{Re[f_1(P_c)^*f_2(P_2)]\nabla_2 e(r_{2c}) - Re[f_1(P_e)^*f_2(P_2)]\nabla_2 e(r_{2e}) \\ & \quad + Re[f_1(P_d)^*f_2(P_2)]\nabla_2 e(r_{2d}) - Re[f_1(P_b)^*f_2(P_2)]\nabla_2 e(r_{2b})\}\cdot d\bar{r}_2 \\ & + \left(\oint_e^c+\oint_b^d\right)\{e(r_{n1})\nabla_1 Re[f_2(P_n)^*f_1(P_1)] - e(r_{f1})\nabla_1 Re[f_2(P_f)^*f_1(P_1)] \\ & \quad + e(r_{g1})\nabla_1 Re[f_2(P_g)^*f_1(P_1)] - e(r_{m1})\nabla_1 Re[f_2(P_m)^*f_1(P_1)]\}\cdot d\bar{r}_1 \\ & + \left(\oint_e^c+\oint_b^d\right)\left(\oint_f^n+\oint_m^g\right) e(r_{21})\nabla_2\{\nabla_1 Re[f_1(P_1)^*f_2(P_2)]\cdot d\bar{r}_1\}\cdot d\bar{r}_2. \end{aligned} \quad (5)$$

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2. J. G. Chaney, 'On the generalized circuit theory as applied to antennas and radiating lines', U. S. N. Postgrad. School Research PaperNo. 1, Mar., 1951

3. J. G. Chaney, 'Free space radiation impedance of rhombic antenna', U. S. N. Postgrad. School Tech. Rpt. no. 4, May, 1952

4. C. F. Klamm, Jr., 'A detailed integration for the radiation impedance of a rhombic antenna', U. S. N. Postgrad. School Tech. Rept. No. 5, June, 1952

It is of interest to examine equation (5) for the three cases in which the two circuits are,

(1) both terminated rhombic antennas,

(2) both open wire antennas,

(3) one rhombic antenna and one open wire antenna.

In each case, not only are the antennas assumed to be symmetrically fed such that the currents are continuous through the generators, but the terminal spacings of the generators are assumed to be negligibly small in comparison with a wave length. Likewise, the terminals for the load impedance of the rhombic antenna are assumed to be negligibly spaced. Then, since the current at the ends of a standing wave antenna vanish, the single integral terms of equation (5) vanish for each of the three cases. Thus,

$$A = (\oint_e^c + \oint_b^d)(\oint_f^n + \oint_m^g)\, e(r_{21}) \nabla_2 \{ \nabla_1 Re[f_1(P_1)^* f_2(P_2)] \cdot d\bar{r}_1 \} \cdot d\bar{r}_2 \tag{6}$$

Substituting into equation (1),

$jkZ_{21}/30 =$

$$\oint_1 \oint_2 e(r_{21}) [\nabla_2 \{ \nabla_1 Re[f_1(P_1)^* f_2(P_2)] \cdot d\bar{r}_1 \} - k^2 Re\{f_1(P_1)^* f_2(P_2)\} d\bar{r}_1] \cdot d\bar{r}_2 \tag{7}$$

In terms of arc lengths along the antennas, let

$$g(ks_1, ks_2) = Re[f_1(P_1)^* f_2(P_2)]$$

$$\cos 2a(s_1, s_2) ds_1 ds_2 = d\bar{r}_1 \cdot d\bar{r}_2$$

and substitute into equation (7),

$$jkZ_{21}/30 = \oint_1 \oint_2 \; e(r_{21}) \; [\frac{\partial^2}{\partial s_2 \partial s_1} - k^2 \cos 2a] \; g(ks_1, ks_2) \; ds_2 ds_1 \tag{8}$$

Equation (8) is in a relatively simple form and is suitable for deriving formulas for the mutual impedances ~~f~~o several antenna configurations. It is true that any formula which requires an *a priori* assumption of the current distribution can be no more than an approximation. However, it has been found that many such formulas are sufficiently accurate for engineering purposes.

OPEN WIRE X-ANTENNA

An interesting application of equation (8) occurs in determining the mutual impedance of the legs of an open wire X-antenna (Figure 2). The conventional sinusoidal current distribution will be assumed to be sufficiently accurate if the legs are not too near antiresonance. Hence,

$$g(ks_1, ks_2) = \sin k(l - |s_1|) \; \sin k(l - |s_2|) \; \csc^2 kl \tag{9a}$$

$$r_{12}^2 = s_1^2 + s_2^2 - 2s_1 s_2 \; \cos 2a \tag{9b}$$

and

$$jkZ_{21}\sin^2 kl/30 =$$

$$\int_{-l}^{+l}\int_{-l}^{+l} e(r_{21})\left[\frac{\partial^2}{\partial s_2\partial s_1} - k^2\cos 2\alpha\right]\sin k(l-|s_1|)\sin k(l-|s_2|)ds_1 ds_2 \tag{10}$$

The integrand has a singularity at the origin. However, one path may be integrated straight through the origin, whereas the other path may be assumed to have a gap of 2ϵ at the origin and the limit taken as ϵ tends to zero. Thus,

$$jkZ_{21}\sin^2 kl/30 =$$

$$\begin{aligned}&\int_{-l}^{0}\int_{-l}^{-\epsilon} e(r_{21})\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\cos 2\alpha\right]\sin k(l+s_1)\sin k(l+s_2)ds_1 ds_2\\
&+\int_{-l}^{0}\int_{\epsilon}^{l} e(r_{21})\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\cos 2\alpha\right]\sin k(l+s_1)\sin k(l-s_2)ds_1 ds_2\\
&+\int_{0}^{l}\int_{-l}^{-\epsilon} e(r_{21})\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\cos 2\alpha\right]\sin k(l-s_1)\sin k(l+s_2)ds_1 ds_2\\
&+\int_{0}^{l}\int_{\epsilon}^{l} e(r_{21})\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\cos 2\alpha\right]\sin k(l-s_1)\sin k(l-s_2)ds_1 ds_2\end{aligned} \tag{11}$$

Let

$$r_{13}^2 = s_1^2 + s_3^2 + 2s_1 s_3\cos 2\alpha \tag{12}$$

and change the variables in equation (11),

$$jkZ_{21}\sin^2 kl/60 =$$

$$\begin{aligned}&\int_{0}^{l}\int_{\epsilon}^{l} e(r_{21})\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\cos 2\alpha\right]\sin k(l-s_1)\sin k(l-s_2)ds_1 ds_2\\
&-\int_{0}^{l}\int_{0}^{l} e(r_{13})\left[\frac{\partial^2}{\partial s_1\partial s_3} + k^2\cos 2\alpha\right]\sin k(l-s_1)\sin k(l-s_3)ds_1 ds_3\end{aligned} \tag{13}$$

After differentiating and transforming trigonometrically,

$$\begin{aligned}Z_{21}\sin^2 kl/jk30 = {}& \sin^2\alpha\,\exp(j2kl)\int_{0}^{l}\int_{\epsilon}^{l} r_{13}^{-1}\exp[-jk(s_1+s_3+r_{13})]ds_1 ds_3\\
&+\sin^2\alpha\,\exp(-j2kl)\int_{0}^{l}\int_{\epsilon}^{l} r_{13}^{-1}\exp[jk(s_1+s_3-r_{13})]ds_1 ds_3\\
&+\cos^2\alpha\int_{0}^{l}\int_{\epsilon}^{l} r_{13}^{-1}\exp[jk(s_1-s_3-r_{13})]ds_1 ds_3\\
&+\cos^2\alpha\int_{0}^{l}\int_{\epsilon}^{l} r_{13}^{-1}\exp[-jk(s_1-s_3+r_{13})]ds_1 ds_3\\
&-\cos^2\alpha\,\exp(j2kl)\int_{0}^{l}\int_{\epsilon}^{l} r_{12}^{-1}\exp[-jk(s_1+s_2+r_{12})]ds_1 ds_2\\
&-\cos^2\alpha\,\exp(-j2kl)\int_{0}^{l}\int_{\epsilon}^{l} r_{12}^{-1}\exp[jk(s_1+s_2-r_{12})]ds_1 ds_2\\
&-\sin^2\alpha\int_{0}^{l}\int_{\epsilon}^{l} r_{12}^{-1}\exp[jk(s_1-s_2-r_{12})]ds_1 ds_2\\
&-\sin^2\alpha\int_{0}^{l}\int_{\epsilon}^{l} r_{12}^{-1}\exp(-jk(s_1-s_2+r_{12})]ds_1 ds_2\end{aligned} \tag{14}$$

Upon comparing the definite integrals of equation (14) with the three integrals evaluated in deriving the self radiation impedance of a rhombic antenna,[3] it is found that the first, second, seventh, and eighth may be written directly, while the third, fourth, fifth, and sixth require only the replacement of the angle α by its complement. It might be mentioned that the value of the integrals do not depend upon the exact form in which the infinitesimal lower limit is taken.

Upon substituting into equation (14),

$$
\begin{aligned}
R_{21}\sin^2 kl/30 &= 2\ln\cot\alpha + 2Ci(2kl\sin\alpha) - 2Ci(2kl\cos\alpha) \\
&+ \sin 2kl\ \{Si[2kl(1+\cos\alpha)]+Si[2kl(1-\cos\alpha)]-Si[2kl(1+\sin\alpha)]-Si[2kl(1-\sin\alpha)]\} \\
&+ \cos 2kl\ \{2\ln\cot\alpha+Ci[(2kl(1+\cos\alpha)]+Ci[2kl(1-\cos\alpha)] \\
&\qquad -Ci[2kl(1+\sin\alpha)]-Ci[2kl(1-\sin\alpha)]\}
\end{aligned}
\tag{15a}
$$

$$
\begin{aligned}
X_{21}\sin^2 kl/30 &= 2Si(2kl\cos\alpha) - 2Si(2kl\sin\alpha) \\
&- \cos 2kl\ \{Si[2kl(1+\cos\alpha) -Si[2kl(1-\cos\alpha)]-Si[2kl(1+\sin\alpha)]+Si[2kl(1-\sin\alpha)]\} \\
&+ \sin 2kl\ \{-4\ln\cot\alpha +Ci[2kl(1+\cos\alpha)]-Ci[2kl(1-\cos\alpha)] \\
&\qquad -Ci[2kl(1+\sin\alpha)]+Ci[2kl(1-\sin\alpha)]\}
\end{aligned}
\tag{15b}
$$

BICONICAL ANTENNA

Equations (15a,b) also give the radiation impedance of a biconical antenna under the sinusoidal current hypothesis, provided α is replaced by $\theta/2$, with 2θ being the apex angle of a nappe of the cone. The logarithmic term in equation (15b) becomes Schelkunoff's biconical characteristic impedance,[5]

$$K_0 = 120\ln\cot\theta/2$$

By taking the apex angle sufficiently small for the current to be almost exactly sinusoidal, the driving point impedance of the biconical antenna may be evaluated from equations (15a,b).. Taking α very near zero and evaluating in the customary manner by writing

$$Si\,\epsilon \approx 0,\ Ci\,\epsilon \approx \ln\gamma\epsilon = C+\ln\epsilon,\quad 0<\epsilon<<1, \tag{17}$$

where C is Euler's constant 0.5772

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5. S.A. Schelkunoff, 'Theory of antennas of arbitrary size and shape', Proc.I.R.E. 29, 9, 493 (1941).

the driving point impedance becomes,

$$\begin{aligned} Z_{in} = \{&60(C + \ln 2kl - Ci2kl) \\ &+30\sin 2kl(Si4kl-2Si2kl) +30\cos 2kl(C+\ln kl+Ci4kl-2Ci2kl) \\ &+j[60Si2kl +30\sin 2kl(Ci4kl-C-\ln kl-2\ln\cot a) -30\cos 2kl\ Si4kl]\} \csc^2 kl \end{aligned} \quad (18)$$

Substituting Schelkunoff's inverse terminal impedance Z_a into equation (18), one obtains[6]

$$Z_{in} = Z_a \csc^2 kl - jK_0 \cot kl \quad (19)$$

which is the value used by Schelkunoff in the evaluation of his inverse terminal impedance, but which was arrived at by a slightly different procedure.[6]

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6. S. A. Schelkunoff, 'Advanced Antenna Theory', John Wiley and Sons, New York, 1952.

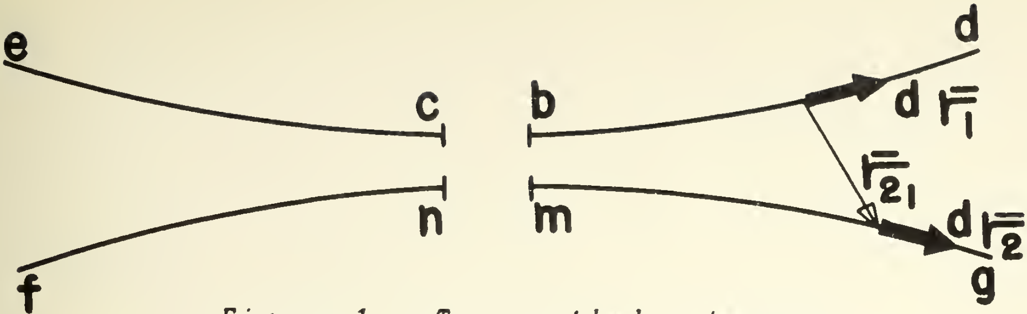

Figure 1. Two coupled antennas.

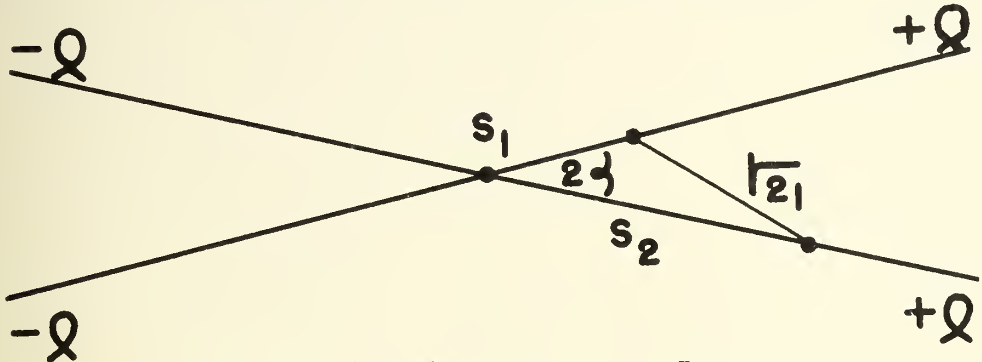

Figure 2. An open wire X-antenna.